

Einstein Equations from Riemann-only Gravitational Actions

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(Dated: January 25, 2013)

In pure affine formalism for gravitation, we construct invariant actions constructed solely from the Riemann tensor. We do this by generalizing the notion of determinant to higher rank tensor fields. In regard to reducing to Einstein-Hilbert action in arbitrary dimensions, we show that Riemann-only actions are more natural than those based on the Ricci tensor. In spacetimes with torsion, we construct a dynamical equation for Riemann tensor itself, and show that it embodies the usual gravitational field equations.

PACS numbers: 04.20.-q, 04.50.Kd, 04.20.Cv, 04.20.Fy

Gravitational field, according to the equivalence principle, is a purely geometrical phenomenon encoded in affine connection Γ which governs parallel transport of tensor fields along a given curve in spacetime. Parallel transport around a closed curve, after one complete cycle, results in a finite mismatch if the spacetime is curved, and curving is uniquely described by the Riemann tensor

$$\mathbb{R}_{\alpha\nu\beta}^{\mu} = \partial_{\beta}\Gamma_{\alpha\nu}^{\mu} - \partial_{\nu}\Gamma_{\alpha\beta}^{\mu} + \Gamma_{\alpha\nu}^{\lambda}\Gamma_{\beta\lambda}^{\mu} - \Gamma_{\alpha\beta}^{\lambda}\Gamma_{\nu\lambda}^{\mu} \quad (1)$$

which is made up solely of the affine connection $\Gamma_{\alpha\beta}^{\lambda}$. This tensor field contracts in two distinct ways to yield the Ricci tensors

$$\begin{aligned} \mathbb{R}_{\alpha\beta}(\Gamma) &= \mathbb{R}_{\alpha\mu\beta}^{\mu}(\Gamma), \\ \overline{\mathbb{R}}_{\alpha\beta}(\Gamma) &= \mathbb{R}_{\mu\alpha\beta}^{\mu}(\Gamma), \end{aligned} \quad (2)$$

wherein $\overline{\mathbb{R}}_{\alpha\beta}(\Gamma)$ does actually turn out to be the antisymmetric part of $\mathbb{R}_{\alpha\beta}(\Gamma)$.

Affine connection determines not only the curving but also the twirling of the spacetime. The latter is encoded in the torsion tensor [1–3]

$$\mathbb{T}_{\alpha\beta}^{\lambda} = \Gamma_{\alpha\beta}^{\lambda} - \Gamma_{\beta\alpha}^{\lambda} \quad (3)$$

which participates in structuring of the spacetime together with curvature tensor. Torsion vanishes in geometries with symmetric connection coefficients.

Having only the curvature and torsion tensors at hand, how far can one impinge to define an invariant action for gravitation? Saying differently, can one construct a consistent geometrodynamical theory by using only the curvature (and maybe also the torsion) tensor? The answer must be affirmative. The reason is that gravitational action, fundamentally, should necessitate no geometrical variable other than connection. In other words, curving and twirling of the spacetime manifold are to be uniquely described by tensor fields generated by connection, and thus, any other geometrical variable, beside connection, can be an extra, non-fundamental ingredient if not a matter or radiation field. Consequently, a general action functional must have the form

$$\mathbb{S} = \int d^D x \, \mathfrak{L}(\Gamma, \partial\Gamma) \quad (4)$$

which involves only the geometrical sector since matter and radiation fields will enter the dynamics by a novel way to be established in the sequel. The action (4) is that of the purely *affine gravity* since its geometrical sector is spanned solely by the connection [1, 2, 4–7]. Obviously, \mathfrak{L} must be a scalar density such that, under general coordinate transformations, changes in \mathfrak{L} must come in a way to compensate the corresponding changes in the differential volume element

$$d^D x = \epsilon_{\mu_0 \mu_1 \dots \mu_{D-1}} dx^{\mu_0} dx^{\mu_1} \dots dx^{\mu_{D-1}} \quad (5)$$

where $\epsilon_{\mu_0 \mu_1 \dots \mu_{D-1}}$ is the permutation symbol. This compensation occurs if \mathfrak{L} involves determinant of some ‘matrix’. Therefore, one naturally writes

$$\mathfrak{L}(\Gamma, \partial\Gamma) \ni (\text{Det} [\mathfrak{G}])^{1/2} \quad (6)$$

where $\mathfrak{G}_{\alpha\beta}$ is a rank (0,2) tensor field, and

$$\text{Det} [\mathfrak{G}] = \frac{1}{D!} \epsilon^{\mu_0 \mu_1 \dots \mu_{D-1}} \epsilon^{\mu'_0 \mu'_1 \dots \mu'_{D-1}} \mathfrak{G}_{\mu_0 \mu'_0} \mathfrak{G}_{\mu_1 \mu'_1} \dots \mathfrak{G}_{\mu_{D-1} \mu'_{D-1}} \quad (7)$$

is its determinant [11]. In general, for fixing the physical content, $\mathfrak{G}_{\alpha\beta}$ can be expressed in terms of various tensor fields of geometrical and material origin [9, 10]. In the geometrical sector, it is highly natural to take

$$\mathfrak{G}_{\alpha\beta} = c_1 \mathbb{R}_{\alpha\beta} + c_2 \overline{\mathbb{R}}_{\alpha\beta} = \frac{c_1}{2} \mathbb{R}_{(\alpha\beta)} + \left(\frac{c_1}{2} + c_2 \right) \overline{\mathbb{R}}_{\alpha\beta} \quad (8)$$

which generalizes the original Eddington proposal [4] by replacing $\mathbb{R}_{(\alpha\beta)}$ by $\mathfrak{G}_{\alpha\beta}$ [12]. The c_1 and c_2 are constants. It is clear that $\mathfrak{G}_{\alpha\beta}$ cannot receive higher-derivative curvature contributions. The reason is that, due to the absence of a metric tensor, curvature tensors cannot be contracted to add higher-derivative structures to $\mathfrak{G}_{\alpha\beta}$.

It is immediately seen that a rank (0,2) tensor field like $\mathfrak{G}_{\alpha\beta}$ is induced also by the torsion tensor. Indeed, direct calculation gives

$$\mathfrak{T}_{\alpha\beta} = c_3 \mathbb{T}_{\mu\nu}^\mu \mathbb{T}_{\alpha\beta}^\nu + c_4 \mathbb{T}_{\mu\alpha}^\mu \mathbb{T}_{\nu\beta}^\nu + c_5 \mathbb{T}_{\nu\alpha}^\mu \mathbb{T}_{\mu\beta}^\nu \quad (9)$$

as a torsion-induced tensor field having the same mass dimension as $\mathfrak{G}_{\alpha\beta}$. Here c_3 , c_4 and c_5 are dimensionless constants. Clearly, $\mathfrak{T}_{\alpha\beta}$ can receive contributions from higher powers of the torsion tensor and its contractions with the curvature tensors. However, all such corrections will be suppressed by the ultraviolet scale \overline{M}_D (which must be around the fundamental scale of gravity in D dimensions). Like $\mathfrak{G}_{\alpha\beta}$, the torsion contribution $\mathfrak{T}_{\alpha\beta}$ also generates a Lagrangian density

$$\mathfrak{L}(\Gamma, \partial\Gamma) \ni (\text{Det} [\mathfrak{T}])^{1/2} \quad (10)$$

similar to (6).

Expectedly, tensor fields like $\mathfrak{G}_{\alpha\beta}$ can also spring from the matter sector: a structure like $\partial_\alpha \varphi \partial_\beta \varphi$ from a scalar field φ , field strength tensor $A_{\alpha\beta}$ of a vector field A_α , and the like [8–10]. Nonetheless, one notes that matter fields can well be incorporated into the formalism not as in (4) by modifying $\mathfrak{G}_{\alpha\beta}$ but as in reference [13] by modifying the connection $\Gamma_{\alpha\beta}^\lambda$. Along with these views, in the following, matter stress-energy tensor will be formulated as dynamically springing from the geometrical sector.

At this point, looking back to (6) and (10), it occurs in mind that the tensor density under concern, \mathfrak{L} , does not have to originate from determinant of a valency-two tensor field like $\mathfrak{G}_{\alpha\beta}$. In fact, for a framework like affine gravity [2, 4, 5, 7], which claims to put gravitational interactions in a more general setting, it is of prime importance to gather all possible sources of interactions. To this end, the structure of (6) can be enriched by incorporating the ‘determinants’ of higher-rank tensor fields [14]

$$\mathfrak{L}(\Gamma, \partial\Gamma) \ni (\text{NDet} [\mathfrak{R}])^{1/N} \quad (11)$$

where \mathfrak{R} is a tensor field of rank N . The function NDet , designating the determinant of a rank- N tensor, reduces to the usual determinant Det in (6) for $N = 2$. As a useful

exemplifying case, it proves enlightening to specialize to a rank (1,3) tensor field $\mathfrak{R}_{\alpha\nu\beta}^\mu$, and write

$$\begin{aligned} \text{DDet} [\mathfrak{R}] &= \frac{1}{(D!)^2} \epsilon^{\mu_0 \dots \mu_{D-1}} \epsilon^{\alpha_0 \dots \alpha_{D-1}} \epsilon^{\nu_0 \dots \nu_{D-1}} \epsilon^{\beta_0 \dots \beta_{D-1}} \\ &\times \mathfrak{R}_{\alpha_0 \nu_0 \beta_0}^{\mu_0} \mathfrak{R}_{\alpha_1 \nu_1 \beta_1}^{\mu_1} \dots \mathfrak{R}_{\alpha_{D-1} \nu_{D-1} \beta_{D-1}}^{\mu_{D-1}} \end{aligned} \quad (12)$$

as its ‘determinant’, more correctly, double-determinant, DDet . An obvious candidate for $\mathfrak{R}_{\alpha\nu\beta}^\mu$ is the Riemann tensor $\mathbb{R}_{\alpha\nu\beta}^\mu$. Another candidate is the Weyl tensor $\mathbb{W}_{\alpha\nu\beta}^\mu$. Any other structure made up of the curvature and torsion tensors would contribute to $\mathfrak{R}_{\alpha\nu\beta}^\mu$ via higher order terms suppressed by the ultraviolet scale \overline{M}_D . Thus, one can write

$$\mathfrak{R}_{\alpha\nu\beta}^\mu = c_6 \mathbb{R}_{\alpha\nu\beta}^\mu + c_7 \mathbb{W}_{\alpha\nu\beta}^\mu \quad (13)$$

where c_6 and c_7 are dimensionless constants. The traceless nature of $\mathbb{W}_{\alpha\nu\beta}^\mu$ guarantees that contractions of $\mathfrak{R}_{\alpha\nu\beta}^\mu$ conform to those of the Riemann tensor $\mathbb{R}_{\alpha\nu\beta}^\mu$, and hence, they must return the Ricci tensors in (2).

The candidate Lagrangian density (11) makes it clear that invariant volumes can be devised via not only the conventional determinant (7) but also the ones like (12). In fact, by combining (6), (10) and (11), one obtains a more general setup

$$\mathcal{S}_{\text{tot}} = \int d^D x \, \mathcal{L}_{\text{tot}} (\Gamma, \partial\Gamma) \quad (14)$$

with the Lagrangian density

$$\mathcal{L}_{\text{tot}} = c_G (\text{Det} [\mathfrak{G}])^{1/2} + c_T (\text{Det} [\mathfrak{T}])^{1/2} + c_R \overline{M}_D^{D/2} (\text{DDet} [\mathfrak{R}])^{1/4} \quad (15)$$

where c_G , c_T and c_I are all dimensionless coefficients. Affine connection is the fundamental dynamical variable; it is the agent which controls each of the terms in (15). Clearly, in forming (15), one could also consider structures involving linear combinations of $\mathfrak{G}_{\alpha\beta}$ and $\mathfrak{J}_{\alpha\beta}$. Physically, however, it suffices focus on the structures in (15).

The formalism structured above, including the candidate invariant actions (6) and (11), involves affine connection $\Gamma_{\alpha\beta}^\lambda$ as the sole geometrical variable. There is no variable other than $\Gamma_{\alpha\beta}^\lambda$. It is the permutation symbol $\epsilon^{\alpha\beta\mu\nu}$ that contracts tensors to generate scalar densities. This geometrical setup, especially the action (14), does neither involve nor necessitate a ‘metric tensor’. In fact, spacetime gets further structured by the notion of distance if it is

endowed with a metric tensor $g_{\alpha\beta}$ embodying clocks and rulers. Actually, connection coefficients and metric tensor are fundamentally independent quantities. They exhibit no *a priori* known relationship. Metric is no more than a device to measure ‘lengths’ and ‘durations’. In this sense, inclusion of metric enhances the geometrodynamical setup (14) in regard to the notions of lengths, durations and angles. The metric tensor can be a ‘built-in geometrical variable’ as in purely metric and metric-affine formalisms or a ‘derived variable’ as in affine gravity [1, 2, 4, 5, 7]. Each option gives a description of the gravitational interactions with specific dynamical correlations and fundamental geometrical variables [9].

For appreciating the physical relevance of (11) in comparison to (6), it proves advantageous to analyze (14) in purely metric formalism by considering a maximally symmetric spacetime, for which

$$\mathbb{R}_{\mu\alpha\nu\beta}(\hat{\Gamma}) = \frac{\kappa}{D(D-1)}(g_{\mu\nu}g_{\alpha\beta} - g_{\alpha\nu}g_{\mu\beta}) \quad (16)$$

where κ is the scalar curvature, $g_{\alpha\beta}$ is the built-in metric tensor on the manifold, and

$$\hat{\Gamma}_{\alpha\beta}^{\lambda} = \hat{\Gamma}_{\beta\alpha}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_{\alpha}g_{\beta\rho} + \partial_{\beta}g_{\rho\alpha} - \partial_{\rho}g_{\alpha\beta}) \quad (17)$$

is the Levi-Civita connection. Torsion vanishes identically. In this setup, the action (14) takes the form

$$\mathbf{S}_{\text{tot}} = \int d^D x (\text{Det}[g])^{1/2} \left\{ \bar{c}_E \kappa^{D/2} + \bar{c}_I \overline{M}_D^{D/2} \kappa^{D/4} \right\} \quad (18)$$

where various dimension-dependent factors arising from reduction are included in the redefinitions of the constants $c_{E,I} \rightarrow \bar{c}_{E,I}$. It could be enlightening to compare action (18) with the Einstein-Hilbert term

$$\mathbf{S}_{\text{EH}} = \int d^D x (\text{Det}[g])^{1/2} \frac{1}{2} \overline{M}_D^{D-2} \kappa \quad (19)$$

as holds in a maximally symmetric spacetime with the curvature tensor (16). Here, it is immediately realized that the Eddington term in (18) agrees with the Einstein-Hilbert action only in $D = 2$ dimensions. In other dimensions, such as $D = 4$, they behave differently. Contrary to the Eddington term, $\mathbf{D}\text{Det}$ term in (18) agrees with the Einstein-Hilbert term only in $D = 4$ dimensions. Consequently, it can be said that, the Riemann-only invariant volume in (11) singles out $D = 4$ as a special dimensionality. In fact, the $\mathbf{N}\text{Det}$ contribution, which, to authors’ knowledge, has never been considered elsewhere, opens up a novel avenue

to generalize the GR by structures distinct from the Einstein-Hilbert term and from various higher-curvature contributions in any of the affine, metric-affine or metric formalisms [1, 7].

In pure affine gravity, by definition, there is no concept of ‘metric’ to speak about. However, whatever the framework is [9], if it has to have a physical content, its dynamical equations must eventually yield the Einstein field equations. In this very sense, it is obligatory to have the notions of ‘metric tensor’ and ‘stress-energy tensor’ emerged in a suitable way. This involves defining these quantities in terms of the partial derivatives of the Lagrangian density [4, 5, 7]. For this purpose, one first notices that the action (14) is an extremum

$$\delta \mathcal{S}_{\text{tot}} = \int d^D x \left(\mathfrak{Q}_\mu^{\alpha\nu\beta} \delta \mathbb{R}_{\alpha\nu\beta}^\mu + \mathfrak{W}_\mu^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu \right) = 0 \quad (20)$$

provided that $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ and $\mathfrak{W}_\mu^{\alpha\beta}$ are related by

$$\nabla_\nu^\Gamma \mathfrak{Q}_\mu^{\alpha\nu\beta} + \mathbb{T}_{\sigma\nu}^\alpha \mathfrak{Q}_\mu^{\sigma\nu\beta} - \frac{1}{2} \mathbb{T}_{\nu\sigma}^\beta \mathfrak{Q}_\mu^{\alpha\nu\sigma} + \mathbb{T}_{\mu\nu}^\sigma \mathfrak{Q}_\sigma^{\alpha\nu\beta} = \mathfrak{W}_\mu^{\alpha\beta} \quad (21)$$

wherein the covariant derivative ∇_α^Γ is that of the connection $\Gamma_{\alpha\beta}^\lambda$ not that of the Levi-Civita connection $\hat{\Gamma}_{\alpha\beta}^\lambda$. From (20) it is self-evident that

$$\mathfrak{Q}_\mu^{\alpha\nu\beta} = \frac{\partial \mathcal{L}_{\text{tot}}}{\partial \mathbb{R}_{\alpha\nu\beta}^\mu} \quad (22)$$

and

$$\mathfrak{W}_\mu^{\alpha\beta} = \frac{\partial \mathcal{L}_{\text{tot}}}{\partial \Gamma_{\alpha\beta}^\mu} \quad (23)$$

are two tensor densities conjugate, respectively, to the Riemann tensor and the affine connection. It is obvious that $\mathfrak{W}_\mu^{\alpha\beta}$ corresponds to hypermomentum (including the contribution of matter sector) studied in [2, 3]. On the other hand, $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ is a new quantity, and its physical meaning will become clear as the analysis proceeds.

The dynamics of $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ is governed by its equation of motion (21). Its solution should return $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ expressed in terms of the affine connection $\Gamma_{\alpha\beta}^\lambda$. Then, through the relation (22), $(\mathfrak{R}^{-1})_\mu^{\alpha\nu\beta}$ and other accompanying tensor densities will be related to $\mathfrak{Q}_\mu^{\alpha\nu\beta}$. This relation then generates a new class of gravitational field equations by expressing $(\mathfrak{R})_{\alpha\nu\beta}^\mu$ in terms of $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ and the other tensor fields present in (22). In summary, by solving (21) and using (22) one expects to obtain a relation of the form

$$\mathbb{R}_{\alpha\nu\beta}^\mu = -\frac{c_7}{c_6} \mathbb{W}_{\alpha\nu\beta}^\mu + \mathfrak{T}_{\alpha\nu\beta}^\mu \quad (24)$$

where $\mathfrak{T}_{\alpha\nu\beta}^\mu$ is a tensor field with the symmetries of Riemann tensor. For physical consistency, its contractions must return the Einstein field equations of gravitation.

For explicating how a relation like (24) arises, it is convenient to discuss the equation of motion (21) in more detail. In particular, it proves advantageous to examine the vanishing torsion limit first. In a torsion-free spacetime, the equation of motion (21) reduces to

$$\nabla_\nu^\Gamma \mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta} = 0 \quad (25)$$

where the superscript $^{(0)}$ designates the condition that $\mathbb{T}_{\alpha\beta}^\lambda = 0$. A physically sensible solution can be obtained by interpreting (25) as implying a certain ‘compatibility’ between $\Gamma_{\alpha\beta}^\lambda$ and $\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta}$. This possibility exhorts one to propose a symmetric ‘metric tensor’ $g_{\alpha\beta}$ (whose matrix inverse is $g^{\alpha\beta}$) with which $\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta}$ develops its structure as $\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta} \propto (\delta_\mu^\nu g^{\alpha\beta} - \delta_\mu^\beta g^{\alpha\nu})$. Direct calculation gives $\text{DDet} [\delta_\mu^\nu g^{\alpha\beta} - \delta_\mu^\beta g^{\alpha\nu}] \propto (\text{Det} [g])^{-1}$, and hence, $\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta}$ obtains the exact form

$$\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta} = \Lambda^{(0)} \sqrt{-g} (\delta_\mu^\nu g^{\alpha\beta} - \delta_\mu^\beta g^{\alpha\nu}) \quad (26)$$

where $\Lambda^{(0)}$ is a parameter of mass dimension $D-2$ in D dimensions. Through this expression, affine connection $\Gamma_{\alpha\beta}^\lambda$ becomes compatible with the metric tensor $g_{\alpha\beta}$. In other words, the affine connection $\Gamma_{\alpha\beta}^\lambda$ is identical to the Levi-Civita connection $\hat{\Gamma}_{\alpha\beta}^\lambda$.

When torsion is non-vanishing, all the terms in the equation of motion (21) get revived. Nevertheless, an affine connection $\Gamma_{\alpha\beta}^\lambda$ compatible with metric $g_{\alpha\beta}$ does still exist as a generalization of the Levi-Civita connection by torsion dependent terms [2, 15]. As expected, the torsion-dependent part is antisymmetric in (α, β) . Thus, metric tensor $g_{\alpha\beta}$ is still allowed to exist, and general solution of (21) shall involve $\mathfrak{Q}_{\mu}^{(0)\alpha\nu\beta}$ as its homogeneous part satisfying (25). Its inhomogeneous part will follow from (21). It will involve torsion-dependent terms as well as $\mathfrak{W}_\mu^{\alpha\beta}$. At this point one immediately notices that the metric tensor alone cannot give a complete parametrization of $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ since new, independent structures are needed for constructing a rank (3,1) tensor density $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ which is endowed with no symmetry other than being antisymmetric in (ν, β) . In this sense, $\mathfrak{Q}_\mu^{\alpha\nu\beta}$ will involve not only $g_{\alpha\beta}$ but also novel tensorial structures not related to $g_{\alpha\beta}$. In any case, the homogeneous solution (26) is vital for establishing the existence of a ‘derived’ metric tensor in the system.

By explicitly computing the derivative of the Lagrangian (15) with respect to the Riemann

tensor one finds for the right-hand side of (22)

$$\begin{aligned}
\text{RHS of (22)} &= \frac{1}{4}c_G (\text{Det} [\mathfrak{G}])^{1/2} \left[(\mathfrak{G}^{-1})^{\alpha\beta} \delta_\mu^\nu - (\mathfrak{G}^{-1})^{\alpha\nu} \delta_\mu^\beta \right] \\
&+ \frac{1}{4}c_R \overline{M}_D^{D/2} (\text{DDet} [\mathfrak{R}])^{1/4} \left[(c_6 + c_7) (\mathfrak{R}^{-1})_\mu^{\alpha\nu\beta} \right. \\
&- \frac{c_7}{D-1} \left((\mathfrak{R}^{-1})_\rho^{\alpha\rho\beta} \delta_\mu^\nu - (\mathfrak{R}^{-1})_\rho^{\alpha\rho\nu} \delta_\mu^\beta \right) \\
&- \left. \frac{c_7(D-3)}{(D^2-1)(D-2)} \delta_\mu^\alpha \left((D-1) (\mathfrak{R}^{-1})_\rho^{\rho\nu\beta} - (\mathfrak{R}^{-1})_\rho^{\nu\rho\beta} + (\mathfrak{R}^{-1})_\rho^{\beta\rho\nu} \right) \right]
\end{aligned} \tag{27}$$

which, at first sight, seems rather complicated. Algebraically, this correct. Physically, however, it is simple in that it involves only the Riemann and Weyl tensors in terms of their matrix inverses, determinants and traces. Therefore, there should occur no loss of generality in choosing the left-hand side of (22) purposefully to obtain a physically-consistent, algebraically-simple relation. Thus it is convenient to write

$$\begin{aligned}
\text{LHS of (22)} &= \frac{1}{4}c_G (\text{Det} [\mathfrak{G}])^{1/2} \left[(\mathfrak{G}^{-1})^{\alpha\beta} \delta_\mu^\nu - (\mathfrak{G}^{-1})^{\alpha\nu} \delta_\mu^\beta \right] \\
&- \frac{1}{4}c_R c_7 \overline{M}_D^{D/2} (\text{DDet} [\mathfrak{R}])^{1/4} \left[\frac{1}{D-1} \left((\mathfrak{R}^{-1})_\rho^{\alpha\rho\beta} \delta_\mu^\nu - (\mathfrak{R}^{-1})_\rho^{\alpha\rho\nu} \delta_\mu^\beta \right) \right. \\
&+ \frac{(D-3)}{(D^2-1)(D-2)} \delta_\mu^\alpha \left((D-1) (\mathfrak{R}^{-1})_\rho^{\rho\nu\beta} - (\mathfrak{R}^{-1})_\rho^{\nu\rho\beta} + (\mathfrak{R}^{-1})_\rho^{\beta\rho\nu} \right) \\
&+ \left. c_t \overline{M}_D^2 (\text{DDet} [t])^{1/4} (t^{-1})_\mu^{\alpha\nu\beta} \right]
\end{aligned} \tag{28}$$

where c_t is a dimensionless constant, and $t_{\alpha\nu\beta}^\mu$ is tensor field to be determined. By equating (28) to (27) one arrives at the promised equation (24) with

$$\mathfrak{T}_{\alpha\nu\beta}^\mu = \frac{1}{\overline{M}_D^2} t_{\alpha\nu\beta}^\mu \tag{29}$$

if one takes

$$c_t = \left(\frac{c_R}{2} \right)^{\frac{4}{(D-4)}}. \tag{30}$$

At this point question is rather clear: What is $t_{\alpha\nu\beta}^\mu$? What is it made up of? What structures can it depend on? There arise many such questions, each revealing certain aspects of $t_{\alpha\nu\beta}^\mu$. First of all, it must be clear from (28) that we have judiciously taken the mass dimension of $t_{\alpha\nu\beta}^\mu$ to be 4. It is for this reason that the factor of $1/\overline{M}_D^2$ arises in (29). The second feature to notice is that we have metric tensor $g_{\alpha\beta}$ at work even with non-vanishing torsion [15]. This has been made clear in obtaining the relation (26) and subsequent discussions. Therefore, $t_{\alpha\nu\beta}^\mu$ must involve the metric tensor. What else can it depend on? Actually, given the number

of independent components of the Riemann tensor $\mathbb{R}_{\alpha\nu\beta}^\mu$ one finds out the necessity of extra structures. In fact, a direct counting reveals that all one needs is a symmetric tensor field, say $\mathcal{T}_{\alpha\beta}$ on top of the metric tensor $g_{\alpha\beta}$. Consequently, as a supplement to the fundamental equation (24), one writes

$$t_{\alpha\nu\beta}^\mu = t_{\alpha\nu\beta}^\mu(g, \mathcal{T}) \quad (31)$$

keeping in mind the relation (29). Obviously, if $\mathcal{T}_{\alpha\beta}$ were replaced by the metric tensor $g_{\alpha\beta}$ then (24) would encode matter-free gravitational field equations since metric tensor is nothing but the stress-energy tensor of vacuum [13]. Saying differently, if one contracts the fundamental equation (24) over, say (μ, ν) indices, it must necessarily reduce to the Einstein field equations for vacuum: $\mathbb{R}_{\alpha\beta} \propto g_{\alpha\beta}$. This is precisely the Eddington-Einstein approach [4, 6, 7].

What is then $\mathcal{T}_{\alpha\beta}$? What is its meaning? The answer is not difficult to locate. If $\mathcal{T}_{\alpha\beta}$ is not identical to the metric tensor $g_{\alpha\beta}$ then it must somehow be related to the stress-energy tensor of matter, radiation and vacuum. In other words, contraction of the fundamental relation (24) over (μ, ν) must give Einstein field equations for gravitation in the presence of matter and radiation. This is the only meaning one can ascribe to $\mathcal{T}_{\alpha\beta}$ if it is to involve tensor structures beyond the metric tensor.

At this point, it could be useful to look back and dwell on the origin of $\mathcal{T}_{\alpha\beta}$ more closely. As mentioned before, the left-hand side of (22), given in (28), is to originate from the equation of motion (21). Except for the torsion-free limit which has revealed the existence of existence of metric tensor through (26), a general solution of (21) is not provided. In other words, the quantity $t_{\alpha\nu\beta}^\mu$ is not explicitly computed from (21). Nevertheless, the expression (28) is proposed to be a general solution of (21), and metric tensor, whose existence is already known from (26), is taken to be within $t_{\alpha\nu\beta}^\mu$ through (31). At first sight this procedure might seem somehow judicious; however, it will be justified by the resulting gravitational field equations.

For revealing the origin of $\mathcal{T}_{\alpha\beta}$, looking from a different angle can prove useful. The solution of (21) in the torsion-free case requires connection $\Gamma_{\alpha\beta}^\lambda$ to be identical to the Levi-Civita connection $\hat{\Gamma}_{\alpha\beta}^\lambda$. Then, in the case of non-vanishing torsion, the connection $\Gamma_{\alpha\beta}^\lambda$ will be generalized by adding a rank (1,2) tensor field $\Delta_{\alpha\beta}^\lambda$ to the Levi-Civita connection. This tensor field represents the general case of non-vanishing torsion. Consequently, $\mathcal{T}_{\alpha\beta}$,

concluded to be related to stress-energy tensor, should be part of $\Delta_{\alpha\beta}^\lambda$. In other words, stress-energy tensor of matter, radiation and vacuum is incorporated in the connection (as was also employed in [13]).

As argued before, for constructing $t_{\alpha\nu\beta}^\mu$ with the same degrees of freedom as the Riemann tensor, one needs two symmetric tensor fields $g_{\alpha\beta}$ and $\mathcal{T}_{\alpha\beta}$. This can be seen from a different angle, too. Indeed, that there must exist another distinct symmetric tensor field is clear from the fact that the Kulkarni-Nomizu product [16] of two symmetric tensors creates an object with the same index symmetries as the Riemann tensor. Therefore, requiring $t_{\alpha\nu\beta}^\mu$ to be first order in $\mathcal{T}_{\alpha\beta}$, one finds that

$$\begin{aligned} t_{\alpha\nu\beta}^\mu &= \frac{1}{D-2} (\delta_\nu^\mu \mathcal{T}_{\alpha\beta} - \delta_\beta^\mu \mathcal{T}_{\nu\alpha} - g_{\alpha\nu} \mathcal{T}_\beta^\mu + g_{\alpha\beta} \mathcal{T}_\nu^\mu) \\ &\quad - \frac{1}{(D-1)(D-2)} g_{\lambda\rho} \mathcal{T}^{\lambda\rho} (\delta_\nu^\mu g_{\alpha\beta} - \delta_\beta^\mu g_{\nu\alpha}) \end{aligned} \quad (32)$$

returns

$$\mathbb{R}_{\alpha\beta} = \frac{1}{M_D^2} \mathcal{T}_{\alpha\beta} \quad (33)$$

upon contracting (μ, ν) in the fundamental equation (24). Comparing with Einstein field equations, one automatically sees role of $\mathcal{T}_{\alpha\beta}$ as being related to stress-energy tensor. Indeed, it must be

$$\mathcal{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} g^{\mu\nu} T_{\mu\nu} \quad (34)$$

if Einstein field equations are to be reproduced correctly. Here, obviously, $T_{\alpha\beta}$ is the conserved stress-energy tensor of matter, radiation and vacuum. A more careful look at (24) and (33) ooze out two dynamical equations:

$$\mathbb{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \mathbb{R}_{\mu\nu} = \frac{1}{M^2} T_{\alpha\beta} \quad (35)$$

and

$$\overline{\mathbb{R}}_{\alpha\beta} = 0. \quad (36)$$

The second equation results dynamically from the symmetric nature of (34). These two equations, (35) and (36), determine thus the two independent contractions (2) of the Riemann tensor in terms of the stress-energy distribution in the medium.

A careful look at the fundamental equation (24) immediately reveals that (32) is of no accident. This follows from the fact that the Kulkarni-Nomizu product (\odot) of two symmetric tensors conforms to the Riemann tensor in regard to index symmetries. In fact, with an eye on the Einstein field equations, the geometrical identity

$$\text{Riemann} = \text{Weyl} + \text{Schouten} \odot \text{metric} \quad (37)$$

conveys the structure in (24) with the Schouten tensor given by [17]

$$S_{\alpha\beta} = \frac{1}{D-2} \left(R_{\alpha\beta} - \frac{1}{2(D-1)} g_{\alpha\beta} R \right) \quad (38)$$

where $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar (formed via the metric tensor). At this point one notices that, the ratio of the coefficients, c_6/c_7 , arising in (24) must be chosen to be -1 for the fundamental equation to agree with (37).

Looking back to (32), one can express $\mathbb{T}_{\alpha\nu\beta}^\mu$ directly in terms of the stress-energy tensor

$$\begin{aligned} \mathbb{T}_{\alpha\nu\beta}^\mu &= \frac{1}{D-2} \left(\delta_\nu^\mu T_{\alpha\beta} - \delta_\beta^\mu T_{\nu\alpha} - g_{\alpha\nu} T_\beta^\mu + g_{\alpha\beta} T_\nu^\mu \right) \\ &\quad - \frac{2}{(D-1)(D-2)} g_{\lambda\rho} T^{\lambda\rho} \left(\delta_\nu^\mu g_{\alpha\beta} - \delta_\beta^\mu g_{\nu\alpha} \right) \end{aligned} \quad (39)$$

by using (34). One notices that, unlike (32) this expression is not the Kulkarni-Nomizu product of $T_{\alpha\beta}$ and $g_{\alpha\beta}$. In other words, $\mathcal{T}_{\alpha\beta}$ is of a rather special form to make the relation (24) complete and balanced in regard to the degrees of freedom involved. One notices that the Weyl tensor in (24) is necessary for the stress-energy tensor $T_{\alpha\beta}$ to be subjected to no constraint other than the conservation of matter and energy [18].

The germinal field equation (24), being a four-index one, involves more degrees of freedom than its contractions (35) and (36). The nature of those extra degrees of freedom is best revealed by (32) which expresses $t_{\alpha\nu\beta}^\mu$ in terms of $\mathcal{T}_{\alpha\beta}$. In view of (33) which relates Ricci tensor to $\mathcal{T}_{\alpha\beta}$, it thus turns out that the dynamics of the aforementioned extra degrees of freedom are contained in the geometrical identity (37). Consequently, the contractions (35) and (36) completely encode the dynamical degrees of freedom, and govern the gravitational dynamics in agreement with the GR.

In light of the analysis above, one concludes that the formalism developed in the present work reproduces the usual gravitational field equations. The interesting aspect of the formalism is that it is the dynamical tensorial equation for Riemann tensor, equation (24), that generates the field equations. Definition of $T_{\alpha\beta}$ in (33) generalizes that of Eddington [4]

to incorporate matter and radiation into the game. Iterating, we have achieved a flavor of the Eddington's approach which yields Einstein's equations in their full generality. The key ingredient we have made use of is the generalization of the concept of determinant which in turn allowed us to write down a Lagrangian density constructed from solely the Riemann tensor. Throughout, there has arisen no need to a metric tensor. The metric as well as the stress-energy tensor, as we have shown, turn out to be derived quantities in accord with the philosophy of the purely affine formalism. They are in a sense buried in the affine connection as implied by the structure of (28).

The present work thus puts forth the novel approach that invariant actions based on Riemann tensor are possible, and they are capable of reproducing the known gravitational dynamics in GR, with plausible assumptions on partial derivations of the action density. The work emphasizes the Riemann tensor, and offers a way of directly putting it on work in defining the gravitational dynamics. As evidenced by form of the action in maximally symmetric spacetimes, Riemann-only gravitational actions could be more plausible in singling out the four-dimensional spacetime as a special dimensionality.

Acknowledgements

Authors would like to thank Victor Tapia for e-mailing them about his work [14] in which determinant of higher-rank tensors was introduced. S. S. thanks Glenn Starkman and Fred Adams for their comments, criticisms and suggestions.

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